

Quantum Kicked Dynamics and Classical Diffusion

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Abstract

We consider the quantum counterpart of the kicked harmonic oscillator showing that it undergoes the effect of delocalization in momentum when the classical diffusional threshold is obeyed. For this case the ratio between the oscillator frequency and the frequency of the kick is a rational number, strictly in analogy with the classical case that does not obey the Kolmogorov-Arnold-Moser theorem as the unperturbed motion is degenerate. A tight-binding formulation is derived showing that there is not delocalization in momentum for irrational ratio of the above frequencies. In this way, it is straightforwardly seen that the behavior of the quantum kicked rotator is strictly similar to the one of the quantum kicked harmonic oscillator, although the former, in the classical limit, obeys the Kolmogorov-Arnold-Moser theorem.

One of the main difficulties facing a theory that tries to understand classical chaos by quantum mechanics is that no diffusional behavior of the quantum model appears. This question was clearly pointed out in [1-2] by using the quantum version of the kicked rotator that, classically, gives rise to the standard map. The latter is the simplest hamiltonian model displaying chaos over a certain threshold. The standard map was generalized in Ref. [3] by considering the effect of a harmonic oscillator potential, originating from a constant magnetic field for a charged particle in a wave packet. The quantized version of this model was studied quite in depth in Ref.[4], where it was clearly showed, by a numerical analysis that no localization appears for some rational value of the kick and the harmonic oscillator frequencies. A generalization of the above results through a dynamic localization model was not however derived.

At the classical level, the kicked harmonic oscillator features two frequencies, the frequency of the oscillator and that of the kick, while the kicked rotator has just the frequency of the kick, but while the latter obeys the Kolmogorov-Arnold-Moser theorem, the former does not since the unperturbed motion is degenerate [3]. Different behaviors appear at a classical level even if both models are known to display chaos. The situation changes at the quantum level as the kicked rotator has the angular momentum quantized as an integer multiple of the Planck's constant. This means that the kicked rotator too has now two frequencies, one being of quantal origin. The main aim of this paper is to show that, at the quantum level, both models display similar behaviors, while are known to differ classically.

The lack of diffusional behavior is what one should expect when a more fundamental theory tries to explain the effects of a less fundamental one [5]. This point was already stressed in Ref. [6]. Here, we derive the diffusional limit of the classical kicked harmonic oscillator by its quantum version and give for it a tight-binding formulation that can show localization for an irrational ratio between the frequencies of the model, exactly as in the case of the quantum kicked rotator. Then, we can conclude, by strict analogy, that the quantum analogue of the diffusional behavior of the classical kicked rotator is the one with a rational ratio of the two frequencies as for the quantum kicked harmonic oscillator.

Let us consider a particle in a weak magnetic field and a wave packet. It

was seen in Ref. [3] that the classical hamiltonian can be cast in the form

$$H = \frac{1}{2}(p^2 + \alpha^2 q^2) - \alpha K \sin(q) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad (1)$$

where the mass of the particle and the wave-number of the kicked potential is set equal to unity. Then, α is the frequency of the oscillator, T the period of the kick and K the strength of the perturbation. We can easily quantize the above hamiltonian by introducing the ladder operators a , a^+ obtaining

$$H = \left(a^+ a + \frac{1}{2}\right) \hbar \alpha - \alpha K \sin(\beta(a^+ + a)) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \quad (2)$$

being now $\beta = \left(\frac{\hbar}{2\alpha}\right)^{\frac{1}{2}}$. For one kick the evolution operator is given by [1-2]

$$U(T^+, 0^+) = e^{\left[\frac{i}{\hbar} \alpha K \sin(\beta(a^+ + a))\right]} e^{\left[-i \alpha T \left(a^+ a + \frac{1}{2}\right)\right]}. \quad (3)$$

Setting $|\psi(0^+) \rangle = \sum_n \psi_n(0^+) |n \rangle$ with $|n \rangle$ the eigenstates of the harmonic oscillator, one gets $|\psi(T^+) \rangle = e^{\left[\frac{i}{\hbar} \alpha K \sin(\beta(a^+ + a))\right]} \sum_n \psi_n(0^+) e^{-i(n+\frac{1}{2})\alpha T} |n \rangle$.

It is easy to see that we get two different behaviors of the wave function depending on whether or not the value $\frac{\alpha T}{2\pi}$ is a rational number. For the main resonant case, $\frac{\alpha T}{2\pi} = 1$, and $|\psi(0^+) \rangle = |0 \rangle$ the ground state of the harmonic oscillator, we arrive at the result

$$|\psi(T^+) \rangle = - \sum_n J_n(z) |in\beta \rangle \quad (4)$$

where $z = \frac{\alpha K}{\hbar}$ and $|in\beta \rangle$ a coherent state with a pure imaginary parameter. It is not difficult to see that this wave function agrees, for certain values of time, with the one of Ref.[6] where a harmonic oscillator in a plane wave was considered. So, in the following we use the same argument of [6] for the computation of the diffusional limit.

It is quite easy to see that, if we turn off the perturbation, since the particle is in its ground state, it has the best possible localization in space and momentum. Turning on the perturbation, while the localization in space is retained, we have

$$\langle p^2 \rangle = p_0^2 (1 + 2\beta^2 z^2) + p_0^2 \sum_{\substack{m,n \\ m \neq n}} [1 + (m+n)^2 \beta^2] J_m(z) J_n(z) e^{-(m-n)^2 \frac{\beta^2}{2}} \quad (5)$$

being $p_0 = \left(\frac{\hbar\alpha}{2}\right)^{\frac{1}{2}}$. With $\beta \gg 1$, at $\sqrt{2}\beta z \sim 1$ the localization in momentum is lost, deviating from the original gaussian result. Then, we easily derive $2\beta^2 z \sim \sqrt{2}\beta \gg 1$ which is the classical diffusional limit, $K \gg 1$, as in Ref. [3]. However, after N kicks, we get $|\psi(NT^+) \rangle = -\sum_n J_n(Nz)|in\beta \rangle$, so, classical diffusion should appear also at small K for large N . This, indeed, happens through a stochastic web [3]. It is also easy to get the result that $\langle p^2 \rangle \propto N^2$ for $N \rightarrow \infty$ as for the quantum kicked rotator.

Now, we show that the above results could be kept for any rational value of the ratio between the oscillator frequency and the kicking frequency. In fact the model can be put in the form of a tight-binding model typical of an electron on a one-dimensional lattice, as also happens for the quantum kicked rotator. By looking at the Floquet eigenstates of the operator U in eq.(3), $U|\psi_\lambda \rangle = e^{-i\lambda}|\psi_\lambda \rangle$, if one sets $|\bar{\psi}_\lambda \rangle = \frac{1}{2} \left[1 + e^{\frac{i}{\hbar}\alpha K \sin(\beta(a^+ + a))} \right] |\psi_\lambda \rangle$, for the probability amplitudes defined through $|\bar{\psi}_\lambda \rangle = \sum_{n=0}^{\infty} c_n |n \rangle$, with $H_0|n \rangle = \left(n + \frac{1}{2}\right) \hbar\alpha |n \rangle$, we arrive at the tight-binding model

$$T_n c_n + \sum_{m \neq n} W_{nm} c_m = \epsilon c_n \quad (6)$$

being

$$T_n = \tan \left[\frac{1}{2} \left(\left(n + \frac{1}{2} \right) \alpha T - \lambda \right) \right] \quad (7)$$

and

$$W_{nm} = \langle n | \tan \left[\frac{\alpha K}{2\hbar} \sin(\beta(a^+ + a)) \right] | m \rangle \quad (8)$$

then, $\epsilon = -W_{nn}$. For this kind of model we have the standard reference [7] where it is shown that, for $\frac{\alpha T}{2\pi}$ an irrational number, eq.(7) is a pseudo-random number generator and we have Anderson localization. For a rational ratio we have delocalized Bloch waves. So, we are arrived at an identical situation as for the quantum kicked rotator. We can conclude that both the quantum kicked harmonic oscillator and the quantum kicked rotator display similar behaviors although the latter, in the classical limit, obeys the Kolmogorov-Arnold-Moser theorem and the former does not. We then also conclude that a correct description of the quantum analog of the classical diffusion in the quantum kicked rotator is given by the case of the rational ratio between the frequencies $\frac{\hbar}{2I}$ of the free motion (I is the moment of inertia) and $\frac{2\pi}{T}$ of the kicks.

Several interesting questions are opened up by the above discussion. We reversed the role of the standard map [1-2] and the standard map with a twist [3] in the quantum limit. However, our analysis requires more study on the quantum kicked harmonic oscillator. In fact, a general Floquet map should be derived in the rational case, as was already done for the quantum kicked rotator [8]. A quantum master equation should be derived for the above cases and the limit $\hbar \rightarrow 0$ taken, proving that the Fokker-Planck equation of the classical case is obtained. We conclude by saying that a lot of new interesting physics could arise from these studies of quantum mechanics, even though quantum theory is now a well known subject, apart from interpretation matters.

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